Li-Yorke Chaos in Quadratic Stochastic Operators

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In 1981, Gy.Targonski formulated a question during the International Symposium in France as to whether there exists a continuous map $f : X \rightarrow X$ of a compact convex manifold $X$, Li-Yorke chaotic, whose scrambled set has positive Lebesgue measure. In one dimensional case, an affirmative answer was given by J. Smital (see [9]). In this paper, we shall affirmatively answer Gy.Targonski’s question in the higher dimensional case. Dynamics of quadratic stochastic Volterra operators (in short Volterra QSO) were deeply studied in [3, 4, 11]. However, we can still observe some chaotic behaviors of such operators. In this paper, we provide a criterion for an existence of a Li-Yorke chaos in a class of Volterra QSO acting on a finite dimensional simplex. Moreover, we shall affirmatively answer Gy.Targonski’s question in the higher dimensional case. In [5], it was given a long self-contained exposition of recent achievements and open problems in the theory of QSO. We refer the reader to [5] for all mentioned notations and conceptions of Volterra QSO.

Let $(X, d)$ be a compact convex manifold and $V : X \rightarrow X$ be a continuous operator. A sequence $\{x^{(n)}\}_{n=0}^{\infty}$, where $x^{(n+1)} = V(x^{(n)})$, $x^{(0)} \in X$, is called a trajectory of $V$ starting from an initial point $x^{(0)}$. Let $Fix(V) = \{x^{(0)} \in X \mid x^{(1)} = x^{(0)}\}$, $Per_k(V) = \{x^{(0)} \in X \mid \exists \ k - \text{smallest} : x^{(k)} = x^{(0)}\}$ be a set of fixed and periodic points, respectively. Let $\omega(x^{(0)})$ be a set of omega limiting points of a trajectory starting from an initial point $x^{(0)}$. An operator $V$ is called regular if the trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ converges for all $x^{(0)} \in X$. It is clear that if $V$ is regular then $Per_k(V) = \emptyset$ for all $k \geq 2$ and $\omega(x^{(0)}) \subset Fix(V)$ with $|\omega(x^{(0)})| = 1$ for all $x^{(0)} \in X$. One of the fascinating results in 1D dynamical system is that if $X = [a, b]$ then $V$ is regular if and only if $Per_k(V) = \emptyset$ for all $k \geq 2$ (see [1]). It is natural to seek an analogy of this incredible result in a higher dimensional case. However, in general, this result does not hold true in the higher dimensional case. As a counter example we can consider the following Volterra QSO $V_{a,b,c} : S^2 \rightarrow S^2$

$$V_{a,b,c} : \begin{cases} x'_1 = x_1(1 - ax_2 + bx_3) \\ x'_2 = x_2(1 + ax_1 - cx_3) \\ x'_3 = x_3(1 - bx_1 + cx_2) \end{cases}$$

where $a, b, c \in [-1, 1] \setminus \{0\}$ and $(x'_1, x'_2, x'_3) \equiv V(x)$.

Let $e_1, e_2, e_3$ be vertices of the simplex $S^2$ and $M_0 = \left(\frac{c}{a+b+c}, \frac{b}{a+b+c}, \frac{a}{a+b+c}\right)$. One can easily check that if $\text{Sign}(a) = \text{Sign}(b) = \text{Sign}(c)$ then $Fix(V_{a,b,c}) = \{e_1, e_2, e_3, M_0\}$, otherwise $Fix(V_{a,b,c}) = \{e_1, e_2, e_3\}$. Moreover, $Per_k(V_{a,b,c}) = \emptyset$ for all $k \geq 2$ and $a, b, c \in [-1, 1] \setminus \{0\}$. However, if $\text{Sign}(a) = \text{Sign}(b) = \text{Sign}(c)$ then the trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ does not converge and the limiting set $\omega(x^{(0)}) \subset \partial S^2$ is infinite for any $x^{(0)} \in int S^2 \setminus \{M_0\}$ (see [3, 4]). This means that $V$ is not regular. Therefore, in the higher dimensional case, an absence of periodic points does not imply the regularity of an operator.
The operator $V_{a,b,c}$ given by (1) is special. This operator is yet another counter example for Ulam’s conjecture. Based on some simulations, S. M. Ulam has conjectured (see [10]) that any QSO acting on a finite dimensional simplex is ergodic, i.e., the first order Cesaro mean $\text{Ces}_1(x^{(0)}, V) = \frac{1}{n+1} \sum_{i=0}^{n} x^{(i)}$ converges for any $x^{(0)} \in S^{m-1}$ and for any QSO $V: S^{m-1} \to S^{m-1}$. It turns out that, in general, Ulam’s conjecture is not true. M. I. Zakharevich showed that the operator $V_{1,1,1}$ given by (1) is not ergodic (see [11]). In a general setting, it was proven [2] that if $\text{Sign}(a) = \text{Sign}(b) = \text{Sign}(c)$ then the operator $V_{a,b,c}$ given by (1) is not ergodic, i.e., the first Cesaro mean $\text{Ces}_1(x^{(0)}, V_{a,b,c})$ does not converge for any $x^{(0)} \in \text{int} S^2 \setminus \{M_0\}$ whenever $\text{Sign}(a) = \text{Sign}(b) = \text{Sign}(c)$. The operator $V_{a,b,c}$ given by (1) has a strange asymptotic behavior. We can see it in the following theorem. We define the $k$-th order Cesaro mean by the following formula

$$\text{Ces}_k(x^{(0)}, V) = \frac{1}{n+1} \sum_{i=0}^{n} \text{Ces}_{k-1}^{(i)} (x^{(0)}, V)$$

where $k \geq 1$ and $\text{Ces}_0(x^{(0)}, V) \equiv x^{(n)}$.

**Theorem 0.1.** Let $V$ be an operator given by (1) and $\text{Sign}(a) = \text{Sign}(b) = \text{Sign}(c)$. Then any order Casaro mean of the trajectory of the operator (1) does not converge, i.e., for any given $k \geq 1$, the $k$-th order Cesaro mean $\text{Ces}_k(x^{(0)}, V_{a,b,c})$ does not converge for any $x^{(0)} \in \text{int} S^2 \setminus \{M_0\}$.

This leads the conclusion to us that the operator $V_{a,b,c}$ has a strange asymptotic behavior whenever $\text{Sign}(a) = \text{Sign}(b) = \text{Sign}(c)$. In fact, we shall show that the operator $V_{a,b,c}$ exhibits the Li-Yorke chaos whenever $\text{Sign}(a) = \text{Sign}(b) = \text{Sign}(c)$.

**Theorem 0.2.** A Volterra QSO given by (1) exhibits the Li-Yorke chaos if and only if $\text{Sign}(a) = \text{Sign}(b) = \text{Sign}(c)$. Moreover, $\text{int} S^2 \setminus \{M_0\}$ is a scrambled set having positive Lebesgue measure.

**References**


