Inverse limits of interval maps: to apply and classify

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Define the Hénon map $H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$:

$$H_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - ax^2 + by \\ x \end{pmatrix}$$

or, if you prefer piecewise linear, the Lozi map $L_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$:

$$L_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - a|\!|x|\!| + by \\ x \end{pmatrix}$$

Global attractors for the Hénon and Lozi maps
Folding Maps of the Plane

For good parameters \((a \in [1, 2], |b| \geq 0 \text{ small})\), the global attractor is

\[
A = \bigcap_{n \geq 0} H_{a,b}^n(U)
\]

for some forward invariant topological disk \(U\).
Folding Maps of the Plane

For good parameters \((a \in [1, 2], |b| \geq 0\) small), the global attractor is

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A = \bigcap_{n \geq 0} H_{a,b}^n(U)
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for some forward invariant topological disk \(U\).

\(A\) is compact, connected and of zero Lebesgue measure (if \(|b| < 1\)), but still very complicated.

It is the closure of the unstable manifold of the saddle fixed point

\[
P = \left( \frac{b+\sqrt{b^2+4a}}{2a}, \frac{b+\sqrt{b^2+4a}}{2a} \right).
\]
Folding Maps of the Plane

**Question 1:** Given only the attactor $A$, can you reconstruct the map $H_{a,b}$?
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**Question 1:** Given only the attactor $A$, can you reconstruct the map $H_{a,b}$?

Locally $A$ looks like a Cantor set of arcs, if you’re not too picky! In reality, there is a lot of substructure making $A$ inhomogeneous.

**Question 1 sharper:** Given only $A$ as topological space (so without embedding in the plane), can you reconstruct the map $H_{a,b}$?
For a continuous map $T : X \to X$ on a compact topological space, the inverse limit is the space of backward orbits:

$$\lim \langle X, T \rangle = \{ \ldots x_{-2}, x_{-1}, x_0 : T(x_{-k-1}) = x_{-k} \in X \ \forall \ k \geq 0 \}.$$
Inverse Limit Spaces

For a continuous map $T : X \to X$ on a compact topological space, the inverse limit is the space of backward orbits:

$$\lim\leftarrow (X, T) = \{ \ldots x_{-2}, x_{-1}, x_0 : T(x_{-k-1}) = x_{-k} \in X \ \forall \ k \geq 0 \}.$$ 

It is equipped with product topology and (now invertible) map

$$\hat{T} : \lim\leftarrow (X, f) \to \lim\leftarrow (X, T)$$

$$(\ldots x_{-2}, x_{-1}, x_0) \mapsto (\ldots x_{-2}, x_{-1}, x_0, T(x_0))$$

A priori, $\hat{T}$ is defined on the Hilbert cube $X^\mathbb{N}$, but embeddings into simpler spaces usually exist.
Folding Maps

For $b = 0,$

$$L_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - a|x| + 0 \cdot y \\ x \end{pmatrix}$$

is non-invertible. Taking $b > 0$ "thickens up" the map, giving each half of a horizontal line two distinct preimage branches.
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This is mimicked by inverse limit spaces of tent maps $T_a(x) = 1 - a|x|$.

Similar for Hénon map $H_{a,b}$ and quadratic map $f_a(x) = 1 - ax^2$. 
Folding Maps

Unimodal inverse limits are simpler than Hénon attractors:

$U$
Folding Maps

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\[ U_{H_{a,b}}(U) \]
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Unimodal inverse limits are simpler than Hénon attractors:

\[ H_{a,b}^n(U) \]

Band of folds (critical values) \{ contracts under \( \hat{f}^n \).

expands and keeps folding under \( H_{a,b}^n \).
Folding Maps

Unimodal inverse limits are simpler than Hénon attractors:

Band of folds (critical values) \[
\begin{cases} 
\text{contracts under } \hat{f}^n. \\
\text{expands and keeps folding under } H_{a,b}^n 
\end{cases}
\]

By a different argument, Barge (1987) showed that in general, a Hénon attractors are not homeomorphic with an inverse limit space of a single one-dimensional bonding map.
Sometimes, however, Hénon attractors are homeomorphic with quadratic inverse limits.

**Theorem (Barge & Holte)**

If \( a \in [1, 2] \) is such that \( f_a(x) = 1 - ax^2 \) has an attracting periodic point, then for \(|b|\) sufficiently small

\[
\lim_{\leftarrow}([1 - a, 1], f_a) \simeq \text{global attractor of } H_{a,b}.
\]
Sometimes, however, Hénon attractors are homeomorphic with quadratic inverse limits.

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*If* $a \in [1, 2]$ *is such that* $f_a(x) = 1 - ax^2$ *has an attracting periodic point, then for* $|b|$ *sufficiently small*

$$\lim_{\rightarrow}([1 - a, 1], f_a) \simeq \text{global attractor of } H_{a,b}.$$  

**Question 2:** Are there different periodic $a, a'$ such that $H_{a,b}$ and $H_{a',b}$ have homeomorphic global attractors?
Inverse limits and hyperbolic maps

Theorem (Williams in 1970s)

For every hyperbolic diffeomorphism $f$ on a manifold $M$, there is a map $g$ on a branched manifold $N$ such that the global attractor of $f$ is homeomorphic with $\lim(M, f)$. 

Example 1: Solenoids within solid torus $g: S^1 \to S^1, x \mapsto 2x \mod 1$.

Example 2: Smale's horseshoe on square with caps $g: [-1, 1] \to [-1, 1], x \mapsto 1 - 2x^2$. 

Solenoid and Horseshoe Map.
Inverse limits and hyperbolic maps

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$$g : [-1, 1] \to [-1, 1], \quad x \mapsto 1 - 2x^2.$$
Inverse limits and hyperbolic maps

**Theorem (Bing ’60, McCord ’65: Classifying Solenoids)**

Let \( g_k : S^1 \to S^1, \ x \mapsto a_k x \pmod{1} \) for an integer sequence \( a_k \geq 2 \), and similar for \( \hat{a}_k \geq 2 \). Then

\[
\lim_{\leftarrow} (S^1, g_k) \simeq \lim_{\leftarrow} (S^1, \hat{g}_k)
\]

if and only if for every prime \( p \)

\[ p | a_k \text{ infinitely often if and only if } p | \hat{a}_k \text{ infinitely often.} \]
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Theorem (Watkins ’82: Classifying Knaster Continua)
Let \( h_r : [0, 1] \to [0, 1] \) be the hat-map with \( r \) branches. Then

\[
\lim_{\leftarrow} ([0, 1], h_r) \simeq \lim_{\leftarrow} ([0, 1], h_{r'})
\]

if and only if

\( r \) and \( r' \) are powers of the same integer.
The Ingram Conjecture

Define the **tent map** $T_s : [0, 1] \rightarrow [0, 1]$ as

$$T_s(x) = \min\{sx, s(1 - x)\}$$

with slope $s \in (1, 2]$, $c_k = T_s^k(\frac{1}{2})$ and core $[c_2, c_1]$.
The Ingram Conjecture

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with slope $s \in (1, 2)$,

$$c_k = T^k_s\left(\frac{1}{2}\right)$$

and core $[c_2, c_1]$.

Conjecture (Ingram’s Conjecture)

*If the slopes $1 \leq s < s' \leq 2$, then*

$$\lim([0, 1], T_s) \not\sim \lim([0, 1], T_{s'}).$$
The Ingram Conjecture - Partial Results

If $c = \frac{1}{2}$ has period $n$ under $T_s$, then $\lim([c_2, c_1], T_s)$ has $n$ endpoints, and elsewhere it is locally a Cantor set of arcs.
The Ingram Conjecture - Partial Results

If \( c = \frac{1}{2} \) has period \( n \) under \( T_s \), then \( \lim([c_2, c_1], T_s) \) has \( n \) endpoints, and elsewhere it is locally a Cantor set of arcs.

Endpoints are topologically distinguishable, so

\[
\lim([c_2, c_1], T_s) \not\cong \lim([c_2, c_1], T_{s'})
\]

if the critical points of \( T_s \) and \( T_{s'} \) have different periods. However, there can still be different slopes achieving the same period. (E.g. there are three slopes for period 5.)
The Ingram Conjecture (IC) - History

Tom Ingram  $\leq 1995$  Attributed it to Stewart Baldwin.
Barge & Diamond  (1995)  IC holds for period 5.
Swanson & Volkmer  (2000)  IC holds for period $\leq 15$.
Kailhofer  (2003)  IC holds for all periods.
Štimac  (2007)  IC holds for all preperiods.
## The Ingram Conjecture (IC) - History

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Further Results

Theorem (Bruin & Štimac (2012))

- For every homeomorphism $g$ of $\lim([0, 1], T_s)$ with slope $s \in (\sqrt{2}, 2]$, there is $R \in \mathbb{Z}$ such that $g$ is isotopic to $\hat{T}^R$.
- The entropy $h_{top}(g) = |R| \log s$.
- If $\text{orb}(c)$ is dense in $[c_2, c_1]$ (and this holds for a.e. slope in $[\sqrt{2}, 2]$), then
  \[ g\mid_{\lim([c_2, c_1], T_s)} = \hat{T}^R. \]
Further Results

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  \[ g|_{\lim([c_2, c_1], T_s)} = \hat{T}^R. \]

Remark: The quadratic family $f_s(x) = sx(1 - x)$ is richer than the tent family, because it allows renormalization (i.e., existence of non-trivial periodic intervals). The Ingram Conjecture for quadratic inverse limits $\lim([0,1], f_s)$ is completely understood, and also the possible values of $h_{\text{top}}(g)$ for homeomorphisms $g$ on $\lim([0,1], f_s)$. 
Subcontinua and Selfsimilarity

A subcontinuum of $X$ is a compact connected proper subset of $X$.

**Theorem (Barge, Brucks & Diamond 1996)**

For a.e. slope $s \in [1, 2]$ and every $t \in [0, 2]$, the inverse limit

$$X = \lim_{\leftarrow}([0, 1], T_s)$$

contains subcontinua $X_t$ that are homeomorphic to $\lim_{\leftarrow}([0, 1], T_t)$.
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The same property (under mild conditions) is true for Hénon attractors and homoclinic tangles.

**Question 3:** Do Hénon attractors contain subcontinua homeomorphic with any other Hénon attractor?
Asymptotic Arc-Components

Two arc-components $C$ and $C'$ are asymptotic if they can be parametrized by $\phi : \mathbb{R} \to C$, $\phi' : \mathbb{R} \to C'$, such that the distance

$$d(\phi(t), \phi'(t)) \to 0 \text{ as } t \to \infty.$$
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Asymptotic Arc-Components

Theorem (Barge & Diamond 1995, Bruin 2005)

- If the critical point $c$ of $T_s$ has periodic $n$, then $\lim \leftarrow (\[0, 1], T_s)$ has between 1 and $2n - 4$ asymptotic arc-components.
- $2n - 4$ is sharp; patterns of arc-components can be computed.
- If $c$ pre-periodic, then there are no asymptotic arc-components.

Question 4: Are there "non-trivial" asymptotic arc-components in $\lim \leftarrow (\[0, 1], T_s)$ if $T_s$ has an infinite critical orbit?

Question 5: Are there asymptotic arc-components in "non-trivial" Hénon attractors?
Asymptotic Arc-Components

Theorem (Barge & Diamond 1995, Bruin 2005)

- If the critical point $c$ of $T_s$ has periodic $n$, then $\lim(\mathbb{I}, T_s)$ has between 1 and $2n - 4$ asymptotic arc-components.
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Question 5: Are there asymptotic arc-components in ”non-trivial” Hénon attractors?
About the Proof of the Ingram Conjecture

Inverse limits $\lim\limits_{\leftarrow}([0, 1], T)$ are chainable: For every $\epsilon > 0$ there is a chain $C = (\ell_j)_{j=1}^N$ such that

$$\lim\limits_{\leftarrow}([0, 1], T) \subset \bigcup_{j=1}^N \text{ and } \ell_j \cap \ell_k \neq \emptyset \text{ iff } |j - k| \leq 1.$$ 

The proof is based on how the arc-component $Z_0$ that contains the endpoint $\bar{0} = (\ldots 0, 0, 0)$, folds through such chains.
About the Proof of the Ingram Conjecture

An arc \( A \subset Z_0 \) is link-symmetric if it passes through \( C \) in a symmetric way:

*The list of indices of the links that \( A \) passes through should be a palindrome.*
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Homeomorphismms map link-symmetric arcs to link-symmetric arcs.
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*The list of indices of the links that $A$ passes through should be a palindrome.*

Homeomorphisms map link-symmetric arcs to link-symmetric arcs.

We study how $\mathbb{Z}_0$ is composed of a concatenation of maximal link-symmetric arcs. Homeos send maximal link-symmetric arcs to maximal link-symmetric arcs.
About the Proof of the Ingram Conjecture

Homeos send $\bar{0}$ to $\bar{0}$. Now the main step is to show that the pattern of concatenated maximal link-symmetric arcs uniquely characterizes $\lim_{\rightarrow}([0, 1], T_s)$.

Question 6: Does the Ingram Conjecture hold for the core $\lim_{\rightarrow}([c_2, c_1], T_s)$?
About the Proof of the Ingram Conjecture

Homeos send $\bar{0}$ to $\bar{0}$. Now the main step is to show that the pattern of concatenated maximal link-symmetric arcs uniquely characterizes $\lim([0, 1], T_s)$.

Thus, in the proof, $Z_0$ is essential. However, we can decompose

$$\lim([0, 1], T_s) = \underbrace{\text{inv. limit}}_{Z_0 \square \underbrace{\text{zero comp.}}_{\lim([c_2, c_1], T_s)}, \underbrace{\text{core}}}_{\text{in which the core is indecomposable (for slopes } s \in (\sqrt{2}, 2)]}$$
About the Proof of the Ingram Conjecture

Homeos send $\tilde{0}$ to $\tilde{0}$. Now the main step is to show that the pattern of concatenated maximal link-symmetric arcs uniquely characterizes $\lim([0, 1], T_s)$.

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**Question 6:** Does the Ingram Conjecture hold for the core $\lim([c_2, c_1], T_s)$?
Fibonacci-like Inverse Limits

Call \( x \in [0, 1] \) a closest precritical point if

\[
\begin{align*}
T^n(x) &= c \quad \text{for some } n \geq 1; \\
T^m(y) &\neq c \quad \text{for } 1 \leq m \leq n, y \in (x, c].
\end{align*}
\]

Such \( n \) is called cutting time. The sequence of cutting times is

\[1 = S_0 < S_1 < \ldots\]
Fibonacci-like Inverse Limits

Call $x \in [0, 1]$ a closest precritical point if

$$\begin{cases} T^n(x) = c & \text{for some } n \geq 1; \\ T^m(y) \neq c & \text{for } 1 \leq m \leq n, y \in (x, c]. \end{cases}$$

Such $n$ is called cutting time. The sequence of cutting times is

$$1 = S_0 < S_1 < \ldots$$

Definition (Fibonacci and Fibonacci-like maps)

Unimodal map $T : [0, 1] \to [0, 1]$ is a Fibonacci map if the cutting times are the Fibonacci numbers:

$$S_0 = 1, \quad S_1 = 2, \quad S_k = S_{k-1} + S_{k-2}.$$ 

$T$ is Fibonacci-like if $S_k - S_{k-1}$ eventually increasing to $\infty$. 


Fibonacci-like Inverse Limits

Theorem (Bruin & Štimac)

If $T$ is a Fibonacci-like tent map, then

$\lim \left( [0, 1], T \right)$ contains uncountably many endpoints.
Fibonacci-like Inverse Limits

Theorem (Bruin & Štimac)

If $T$ is a Fibonacci-like tent map, then

- $\lim_{\leftarrow}([0, 1], T)$ contains uncountably many endpoints.
- A proper subcontinuum of $\lim_{\leftarrow}([c_2, c_1], T)$ is homeomorphic with a point, arc or $\sin \frac{1}{x}$-continuum. No subcontinua more complicated than those.

A $\sin \frac{1}{x}$-continuum is the closure of the graph of $\sin \frac{1}{x}$ on $(-\infty, 0)$.
Fibonacci-like Inverse Limits

Theorem (Bruin & Štimac)

If $T$ is a Fibonacci-like tent map, then

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A sin $\frac{1}{x}$-continuum is the closure of the graph of $\sin \frac{1}{x}$ on $(-\infty, 0)$

- The core Ingram conjecture holds for $\lim (\left[c_2, c_1\right], T)$. 

The Ingram Conjecture - Revised History

- **Barge & Diamond** core IC holds for period 5.
- **Swanson & Volkmer** core IC holds for period \( \leq 15 \).
- **Kailhofer** core IC holds for all periods.
- **Štimac** core IC holds for all preperiods.
- **Barge, Bruin & Štimac** IC holds for all slopes.

Some day we will prove the general Core Ingram Conjecture, but the real challenge is to classify Hénon-like attractors.
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Bruin & Štimac (2013) core IC holds for all Fibonacci-likes.

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# The Ingram Conjecture - Revised History

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